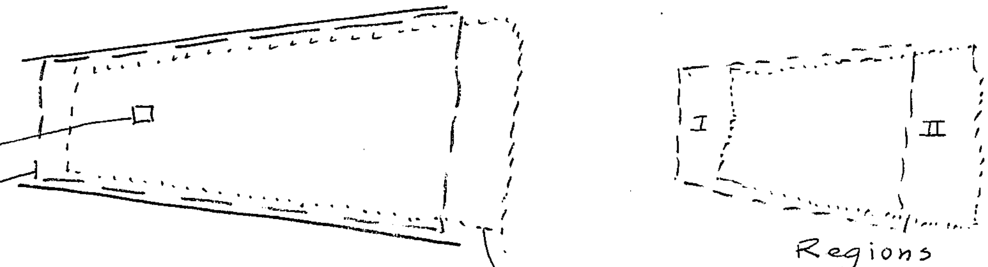
REYNOLDS TRANSPORT THEOREM
(RTT)

Let $B_{sys}(t)$ = amount of physical parameter present in system at time t.

$B_{sys}(t+\delta t)$ = same, but δt later

where B is some 'extensive' property

Let $b = \dfrac{B}{m}$ ('intensive' property)



Fixed control surface
And system boundary at t.

System boundary at $t+\delta t$

consider an elemental volume, dB_{sys}

$$dB_{sys} = b\,dm$$
$$= b\rho\, d\forall$$

integrate to find B_{sys}:

$$\int dB_{sys} = \int b\rho\, d\forall \qquad \text{(Note: volumetric integral)}$$

$$\Rightarrow \quad B_{sys} = \int b\rho\, d\forall \qquad \text{(use later)}$$

How do we keep track of our system as it moves out (and more comes in) of the control volume?

13-782 500 SHEETS FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE 5 SQUARE
42-382 100 SHEETS EYE-EASE 5 SQUARE
42-389 200 SHEETS EYE-EASE 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
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From 'Regions' sketch,

$\forall_{II}$ = outflow of system in δt (+)

$\forall_{I}$ = inflow of system in δt (−)

$\forall_{cv} = \forall_{cv}$ = volume of control volume = fixed

From the drawing,

$$B_{sys}(t) = cV = B_{cv}(t)$$

$$B_{sys}(t+\delta t) = (cV - I) + II$$

$$= B_{cv}(t+\delta t) - B_{I}(t+\delta t) + B_{II}(t+\delta t)$$

Now in general, change of B_{sys} in δt is:

$$\frac{\delta B_{sys}}{\delta t} = \frac{B_{sys}(t+\delta t) - B_{sys}(t)}{\delta t}$$

$$= \frac{B_{cv}(t+\delta t) - B_{I}(t+\delta t) + B_{II}(t+\delta t) - B_{sys}(t)}{\delta t}$$

But recall that $B_{sys}(t) = B_{cv}(t)$ so collect common terms from last equation:

$$\underset{①}{\frac{\delta B_{sys}}{\delta t}} = \underset{②}{\frac{B_{cv}(t+\delta t) - B_{cv}(t)}{\delta t}} \underbrace{- \frac{B_{I}(t+\delta t)}{\delta t} + \frac{B_{II}(t+\delta t)}{\delta t}}_{③}$$

This is embryonic form of RTT:

① $\frac{\delta B_{sys}}{\delta t}$ as $\delta t \to 0 = \frac{D B_{sys}}{Dt}$

② is how B_{sys} changes inside c.v. in time

③ is how much B_{sys} enters and leaves control volume

Now: look at term ②:

$$\lim_{\delta t \to 0} \frac{B_{cv}(t+\delta t) - B_{cv}(t)}{\delta t} = \frac{\partial B_{cv}}{\partial t} = \frac{\partial \int \rho b \, d\forall}{\partial t}$$

integral from bottom of page 1

Now: look at term ③:

B_{II} represents 'how much' B has left the C.V.:

$B_{II} = b_2 m_2 = b_2 \rho_2 \delta\forall_{II}$ in general

but $\delta\forall_{II}$ is a volume equal to $A_2 \delta l_2$ where δl_2 is length traveled in δt, equal to $V \delta t$

$\Rightarrow \delta\forall_{II} = A_2 (V_2 \delta t)$. So

so $B_{II}(t+\delta t) = b_2 \rho_2 A_2 V_2 \delta t$

now to get a flow rate, divide by δt and take a limit:

$$\lim_{\delta t \to 0} \frac{B_{II}(t+\delta t)}{\delta t} = \dot{B}_{OUT} = b_2 \rho_2 A_2 V_2$$

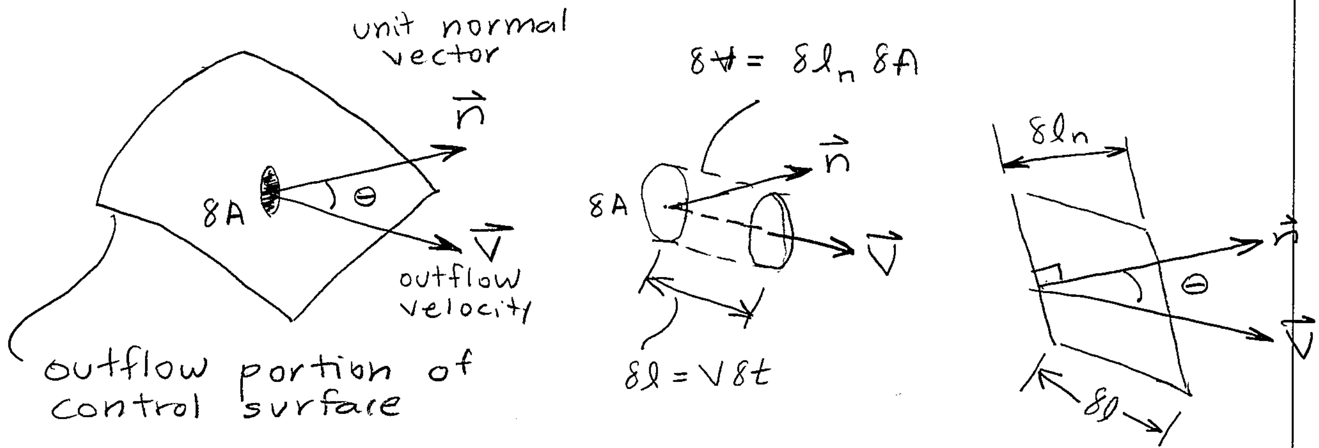
where $\dot{B}_{OUT}$ — "rate" — leaving CV.

similarly, $\dot{B}_{IN} = b_1 \rho_1 A_1 V_1$

so putting it all together,

$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} + \dot{B}_{OUT} - \dot{B}_{IN}$$

Now - allow for B to enter and leave C.V. at an angle θ:



From above geometry, $\vec{V}$ makes angle θ with a unit vector normal to surface.

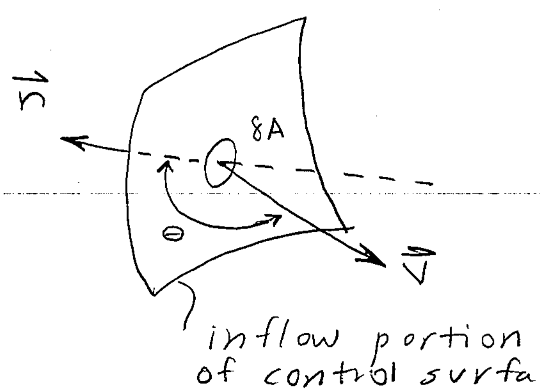
thus $\delta \dot{B}_{OUT} = \rho b V \cos\theta \, \delta A$

and $\int \delta \dot{B}_{OUT} = \int \rho b V \cos\theta \, dA$

$\Rightarrow \dot{B}_{OUT} = \int_{CS_{OUT}} \rho b \vec{V} \cdot \vec{n} \, dA \qquad (V\cos\theta = \vec{V} \cdot \vec{n})$

this is how much B leaves CV at any angle

Now for inflow,



by definition, $\vec{n}$ is always outward from surface

so we'll have a negative sign

$\Rightarrow \dot{B}_{IN} = - \int_{CS_{IN}} \rho b \vec{V} \cdot \vec{n} \, dA$

AND $\dot{B}_{OUT} - \dot{B}_{IN} = \int_{CS_{TOTAL}} \rho b \vec{V} \cdot \vec{n} \, dA$

SO FINALLY,

$$\underbrace{\frac{DB_{sys}}{Dt}}_{\text{how system changes in space and time}} = \underbrace{\frac{\partial}{\partial t}\int \rho b \, d\forall}_{\text{how system changes in time } \underline{\text{inside}} \text{ c.v.}} + \underbrace{\int_{cs} \rho b \vec{V} \cdot \vec{n} \, dA}_{\text{how much system leaves and enters c.v. in time}}$$

$$\frac{DB_{sys}}{Dt} = \text{"local"} + \text{"convective"}$$

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