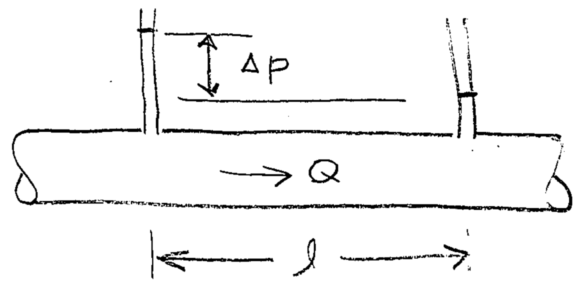
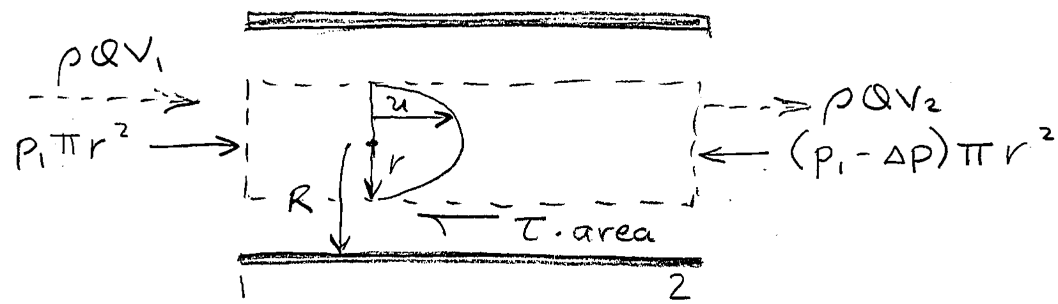
DERIVATION OF HAGEN-POISEUILLE EQN

GIVEN: Laminar flow in a straight, horizontal pipe of radius R_1 and length l

PROVE: $Q = \dfrac{\pi D^4 \Delta p}{128 \mu l}$

SOLUTION:





assume: ρ constant and incompressible
$\vec{V} \perp$ to control surfaces
2 control surfaces
fluid is <u>NOT</u> inviscid

$$\Rightarrow \quad \Sigma F_x = \rho Q \left[\int_{OUT} u(r)\, dA - \int_{IN} u(r)\, dA \right]$$

BUT SINCE $r_1 = r_2$ AND $Q_1 = Q_2$, $\bar{V}_1 = \bar{V}_2$, SO

$$\Sigma F_x = 0!$$

$$\Sigma F = p_1 \pi r^2 - (p_1 - \Delta p)\pi r^2 - \tau \cdot \text{area}$$

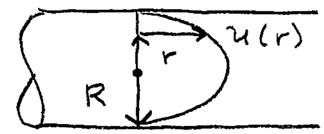
where area =

$$\Rightarrow \quad \Delta p \, \pi r^2 = \tau \, (2 \pi r \ell)$$

$$\Rightarrow \quad \frac{\Delta p}{\ell} = \frac{2\tau}{r}$$

↑

what we want to know

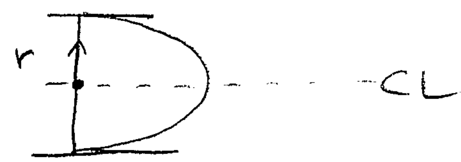


So... detour to look at τ at the wall

since $\frac{\Delta p}{\ell} \neq \phi(r)$, $\Rightarrow \frac{2\tau}{r} \neq \phi(r)$ either

$$\Rightarrow \quad \tau = \frac{\Delta p}{\ell} \frac{r}{2} = Cr \quad \text{where} \quad C = \frac{\Delta p}{\ell \cdot 2} \qquad (1)$$

Now looking at $u(r)$,



when $r = 0$, $\tau = 0$

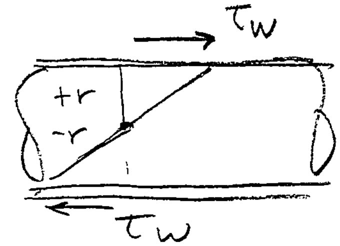
$r = R$, $\tau = \tau_{wall} = \tau_{max}$

$$\Rightarrow \quad \tau_w = CR = \frac{CD}{2} \quad \text{where } D = \text{diameter}$$

and $\quad C = \frac{2\tau_w}{D}$

plugging into (1), $\quad \underline{\underline{\tau(r) = 2\tau_w \frac{r}{D}}}$

τ distribution is linear...



What about velocity profile?

From $\tau = \mu \frac{du}{dy}$, for our case, $dy = dr$

as $r \uparrow$, $u \downarrow$ so we have

$$\tau = -\mu \frac{du}{dr} \quad \text{to make } \tau \text{ positive}$$

from $\frac{\Delta p}{\ell} = \frac{2\tau}{r}$,

and $\frac{\Delta p}{\ell} = \frac{2}{r}\left(-\mu \frac{du}{dr}\right)$ NOW <u>INTEGRATE</u>!

$$\int du = -\int \frac{\Delta p}{\ell} \frac{1}{2\mu} r\,dr$$

$$u = \frac{-\Delta p\, r^2}{4\mu\ell} + C$$

To evaluate C, recognize that when

$r = R = \frac{D}{2}$, $u = 0$

$$\Rightarrow \quad 0 = \frac{-\Delta p R^2}{4\mu\ell} + C$$

$$\Rightarrow \quad C = \frac{\Delta p}{4\mu\ell}\frac{D^2}{4} = \frac{\Delta p D^2}{16\mu\ell}$$

and $$u(r) = \frac{-\Delta p}{4\mu\ell} r^2 + \frac{\Delta p D^2}{16\mu\ell} \qquad (2)$$

now when $r = 0$, $u = u_{max} = V_c$ (define)

$$V_c = \frac{\Delta p D^2}{16\mu\ell}$$

work this into (2)

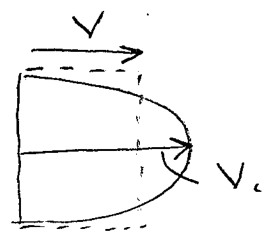
Take (2), reverse order on right-hand side:

$$u(r) = \frac{\Delta p D^2}{16 \mu l} - \frac{\Delta p r^2}{4 \mu l} \cdot \underbrace{\frac{D^2}{D^2} \cdot \frac{4}{4}}_{\text{getting so we can use } V_c}$$

$$u(r) = \underbrace{\frac{\Delta p D^2}{16 \mu l}}_{V_c} - \underbrace{\frac{\Delta p D^2}{16 \mu l}}_{V_c} \left(\frac{2r}{D}\right)^2$$ "leftover"

$$u(r) = V_c \left[1 - \left(\frac{2r}{D}\right)^2 \right]$$ velocity distribution

and average velocity?



$Q = V \cdot A$ where V = avg velocity

$$\int V \, dA = \int u(r) \, dA$$

$$= \int u(r) \cdot 2\pi r \, dr$$

$dA = 2\pi r \, dr$

$$V \cdot \frac{\pi D^2}{4} = \int_0^R V_c \left[1 - \left(\frac{2r}{D}\right)^2 \right] 2\pi r \, dr$$

$$= 2\pi V_c \int_0^R \left[1 - \left(\frac{2r}{D}\right)^2 \right] r \, dr$$

$$= 2\pi V_c \left| \frac{r^2}{2} - \frac{4}{D^2} \frac{r^4}{4} \right|_0^R$$

$$V \cdot \frac{\pi D^2}{4} = 2\pi V_c \left(\frac{R^2}{2} - \frac{R^4}{D^2} \right)$$

or $$V \cdot \frac{\pi D^2}{4} = 2\pi V_c \left(\frac{D^2}{8} - \frac{D^2}{16} \right) \quad \text{from } R = \frac{D}{2}$$

$$V \frac{\pi D^2}{4} = 2\pi V_c \frac{D^2}{16}$$

or $$V = \frac{V_c}{2} \; !$$

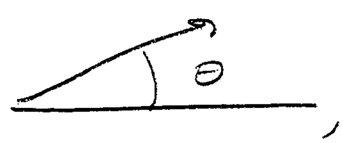
AND $$Q = V \cdot \frac{\pi D^2}{4} = \frac{1}{2}(V_c)\frac{\pi D^2}{4}$$

$$= \frac{1}{2} \frac{\Delta p D^2}{16 \mu l} \cdot \frac{\pi D^2}{4}$$

$$Q = \frac{\pi D^4 \Delta p}{128 \mu l} \qquad \text{Q.E.D.}$$

$$V = \frac{\Delta p D^2}{32 \mu l}$$

and considering flow at an angle θ



$$Q = \frac{\pi (\Delta p - \gamma l \sin\theta) D^4}{128 \mu l}$$

$$V = \frac{(\Delta p - \gamma l \sin\theta) D^2}{32 \mu l}$$